

HIDDEN SYMMETRIES AND CATEGORICAL REPRESENTATION THEORY.

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ABSTRACT. The interrelations between the inverse problems of the representation theory and the categorical representation theory are discussed.

This short article is devoted to the interrelations between the inverse problems of the representation theory (see [1-4]) and the categorical representation theory. The necessity of their analysis was clearly explicated in the papers [3,4], where some non-standard representation theories appeared. It seems that the machinery of the categorical representation theory is very natural to work with some inverse problems of the representation theory. Ideologically, it is truly remarkable that an analysis of concrete inverse problems allows to enlight “dark corners” of the foundations of the theory.

The article is organized as a sequence of topics just as the previous ones [2,3] and may be considered as their continuation devoted to a specific subject. However, the concrete topics are prefaced by an introduction, in which a general formalism of the categorical representation theory is exposed. The paper contains three technically simple but important theorems with straightforward and self-evident proofs, which are, therefore, omitted.

INTRODUCTION: ELEMENTS OF THE CATEGORICAL REPRESENTATION THEORY

We shall consider the representations of classes of objects, which constitute a category, which will be called *the ground category*. The categorical aspects of the standard representation theory were discussed in [5]. Some categorical generalizations were described in [6]. However, we shall formulate the most abstract settings, which are necessary for our purposes.

Definition 1A. A *representation theory* for the ground category \mathcal{A} is a contravariant functor R from the category \mathcal{A} to the category \mathcal{ABEL} of all small abelian categories.

Sometimes one should consider the category \mathcal{ADD} of all small additive categories instead of \mathcal{ABEL} . However, we shall consider the least category for simplicity.

Often the ground category has some good properties, e.g. that for any finite family of objects there exists their coproduct, which coincides with their product. Such situation is realized for Lie algebras, Lie groups, finite groups, associative algebras, Hopf algebras and many other structures. However, the isotopic pairs (see f.e. [1:§2.2]) and the most of other algebraic pairs do not form a category of such type. For the ground category \mathcal{A} , in which products and coproducts of finite number of objects exist and coincide, we shall claim in the definition of representation theories that an associative family of imbeddings $R(a) \times R(b) \hookrightarrow R(a + b)$ (a and b are any objects of the ground category \mathcal{A}) is defined. Such representation theories will be called *quasitensorial*.

Remark 1. If an object a of the ground category \mathcal{A} admits a coassociative monomorphism ε into $a + a$ then $R(a)$ is a tensor category iff the representation theory R is quasitensorial.

There exist non-quasitensorial representation theories even for the well-known categories of the represented objects, e.g. general \mathcal{HS} -projective representations of Lie algebras [3] or unitary \mathcal{HS} -pseudorepresentations of Lie groups [4] are out of this class.

Definition 1B. A representation theory for the ground category \mathcal{A} is called *homomorphic* iff there exists a subcategory \mathcal{A}_0 of \mathcal{A} (*the target subcategory*) such that for any object a of \mathcal{A} the category $R(a)$ may be identified with the category $\text{Mor}(a, \mathcal{A}_0)$ of all (equivalence classes of) morphisms from a to the objects of the category \mathcal{A}_0 .

For instance, theories of all linear, projective, unitary representations of Lie groups are homomorphic. Note that the target category \mathcal{A}_0 is always an additive subcategory of the ground category \mathcal{A} .

Definition 1C. A representation theory for the ground category \mathcal{A} is called *hiddenly homomorphic* iff there exists a homomorphic representation theory R' for a category \mathcal{K} and a functor (multi-valued as a rule) $\varrho : \mathcal{A} \mapsto \mathcal{K}$ such that $R = R' \circ \varrho$.

Below we shall consider some examples and general constructions of the hiddenly homomorphic representation theories for the ground category \mathcal{LIE} of the Lie algebras, which are not homomorphic, and describe their interpretations in terms of the categorical representation theory, thus, providing the least by the further elaboration of its details.

TOPIC ONE: THE COMPOSITE REPRESENTATION THEORIES

Definition 2 [3].

A. A linear space \mathfrak{v} is called a *Lie composite* iff there are fixed its subspaces $\mathfrak{v}_1, \dots, \mathfrak{v}_n$ ($\dim \mathfrak{v}_i > 1$) supplied by the compatible structures of Lie algebras. Compatibility means that the structures of the Lie algebras induced in $\mathfrak{v}_i \cap \mathfrak{v}_j$ from \mathfrak{v}_i and \mathfrak{v}_j are the same. The Lie composite is called *dense* iff $\mathfrak{v}_1 \uplus \dots \uplus \mathfrak{v}_n = \mathfrak{v}$ (here \uplus means the sum of linear spaces). The Lie composite is called *connected* iff for all i and j there exists a sequence k_1, \dots, k_m ($k_1 = i$, $k_m = j$) such that $\mathfrak{v}_{k_l} \cap \mathfrak{v}_{k_{l+1}} \neq \emptyset$.

B. A *representation* of the Lie composite \mathfrak{v} in the space H is the linear mapping $T : \mathfrak{v} \mapsto \text{End}(H)$ such that $T|_{\mathfrak{v}_i}$ is a representation of the Lie algebra \mathfrak{v}_i for all i .

C. Let \mathfrak{g} be a Lie algebra. A linear mapping $T : \mathfrak{g} \mapsto \text{End}(H)$ is called *the composed representation* of \mathfrak{g} in the linear space H iff there exists a set $\mathfrak{g}_1, \dots, \mathfrak{g}_n$ of the Lie subalgebras of \mathfrak{g} , which form a dense connected composite and T is its representation.

Reducibility and irreducibility of representations of the Lie composites are defined in the same manner as for Lie algebras. One may also formulate a superanalog of the Definition 1. The set of representations of the fixed Lie composite is closed under the tensor product and, therefore, may be supplied by the structure of a tensor category. The theory of the composed representations is evidently nonhomomorphic but hiddenly homomorphic theory.

The examples of the Lie composites and their representations were exposed in [3]. The composed representations of the Witt algebra by the tensor operators of spin 2 (q_R -conformal symmetries) in the Verma modules over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ were constructed in [3], too. They are representations of the dense connected Witt composite and generate a tensor subcategory in the category of all composed representations of the Witt algebra.

Below we shall formulate the general categorical setting for the construction of the composed representations.

Definition 3A. Let \mathcal{A} be a topologized ground category (i.e. supplied by a structure of the Grothendieck topology [7]). Let R be a representation theory for \mathcal{A} . *The composed representation theory* $\mathbf{C}(R)$ for \mathcal{A} may be constructed in the following manner. Let a be an object of the ground category \mathcal{A} and $S = (s_1, s_2, \dots, s_n)$ ($s_i \in \text{Mor}(a_i, a)$) be a cover of a then the objects of the category $\mathbf{C}(R)(a)$ consists of all data (b_1, b_2, \dots, b_n) , $b_i \in R(a_i)$ such that for any object c and monomorphisms $f \in \text{Mor}(c, a)$ and $f_i \in \text{Mor}(c, a_i)$ ($f = s_i \circ f_i$) the equality (*the composite glueing rule*)

$$R(f_i)^*(b_i) = R(f_j)^*(b_j)$$

holds. The morphisms in $\mathbf{C}(R)(a)$ are defined in the same manner.

For any representation theory R the composite representation theory $\mathbf{C}(R)$ is a sheaf of abelian categories over the topologized ground category \mathcal{A} [7]. It is a sheaf canonically constructed from the pre-sheaf R over the topologized ground category \mathcal{A} (note that the representation theory for the topologized ground category \mathcal{A} is just a pre-sheaf over it).

Theorem 1. *The composed representations of Lie algebras form a composed representation theory $\mathbf{C}(R)$, where R is a standard representation theory of Lie algebras (the covers of the Lie algebras are defined by the dense connected Lie composites).*

Note that the Grothendieck topology of the Theorem 1 differs from the usual one.

Remark 2. If R is the standard representation theory then the theory $\mathbf{C}(R)$ is hiddenly homomorphic, the category \mathcal{K} is one of the Lie composites, the category \mathcal{K}_0 consists of Lie algebras $\text{End}(H)$ for all linear spaces H . i.e. just the same as for a homomorphic standard representation theory. However, if R is a general representation theory $\mathbf{C}(R)$ is not obligatory hiddenly homomorphic.

I suspect that the concept of the hidden homomorphicity of the composite representation theories may be somehow understood in terms of the topos theory [7].

Remark 3. $\mathbf{C}(\mathbf{C}(R)) = \mathbf{C}(R)$.

TOPIC TWO: THE OVERLAY REPRESENTATION THEORIES

The main disadvantage of the composed representation theory is clearly explicated on the examples of the composed representations of the Witt algebras by the hidden infinite dimensional (q_R -conformal) symmetries in the Verma modules over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. First, the tensor product of a finite number of these irreducible composed representations is irreducible. This fact contradicts to the naïve intuition. Second, the hidden symmetries do not form any representation themselves whereas intuitively they should form the adjoint representation.

So one needs some generalization of the composed representations. Let us define the operator Lie composites $\text{LC}(H)$ as the sets of subspaces $\text{End}(H_i)$ in the spaces $\text{End}(H)$ ($H = H_1 + \dots + H_m$) with the natural structures of Lie algebras.

Definition 4.

A. An *overlay representation* of the Lie composite \mathfrak{v} in the space H is the homomorphism T of \mathfrak{v} into the operator Lie composite $\text{LC}(H)$.

B. Let \mathfrak{g} be a Lie algebra. A linear mapping $T : \mathfrak{g} \mapsto \text{End}(H)$ is called *the overlay composed representation* (or simply *overlay representation*) of \mathfrak{g} in the linear space H iff there exists a set $\mathfrak{g}_1, \dots, \mathfrak{g}_n$ of the Lie subalgebras of \mathfrak{g} , which form a dense connected composite and T is its overlay representation.

Remark 4. The overlay representations of any Lie algebra \mathfrak{g} form a tensor category.

The overlay representations solve the previously described difficulties.

Theorem 2. *The tensor operators of spin 2 in the Verma modules V_h over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ form an overlay representation of the Witt algebra, which are subrepresentations of $\text{End}(V_h)$.*

Remark 5. The tensor operators of any natural spin n in the Verma modules V_h over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ form overlay representations of the Witt algebra, which are subrepresentations of $\text{End}(V_h)$.

Let us formulate the natural categorical settings for the construction of overlay representations.

Definition 3B. Let \mathcal{A} be a topologized ground category. Let R be a homomorphic representation theory for \mathcal{A} with the target subcategory \mathcal{A}_0 supplied by the Grothendieck topology induced from \mathcal{A} . *The overlay representation theory* $\mathbf{O}(R)$ for \mathcal{A} may be constructed in the following manner. Let a be an object of the ground category \mathcal{A} and $S = (s_1, s_2, \dots, s_n)$ ($s_i \in \text{Mor}(a_i, a)$) be a cover of a then the objects of the category $\mathbf{O}(R)(a)$ consists of all data (r_1, r_2, \dots, r_n) , $r_i \in \text{Mor}(a, b_i)$, b_i are objects of the target subcategory \mathcal{A}_0 , which form a cover of the object b of the same subcategory by the monomorphisms $t_i \in \text{Mor}(b_i, b)$, such that for any subobject $(c; p)$ of a ($p \in \text{Mor}(c, a)$) the equality (*the overlay glueing rule*)

$$r_i((a_i; s_i] \cap (c; p]) \cap (b_j; t_j] = (b_i; t_i] \cap r_j((a_j; s_j] \cap (c; p])$$

holds. The morphisms in $\mathbf{O}(R)(a)$ are defined in the same manner.

However, I do not know a definition of the overlay representation theory $\mathbf{O}(R)$ for the representation theory R , which is not homomorphic. Note that $\mathbf{O}(R)$ is not a sheaf of abelian categories over \mathcal{A} in general, and I do not know an abstract sheaf theoretical characterization of the overlay representation theories.

Remark 6. The overlay representation theories $\mathbf{O}(R)$ being defined for the homomorphic representations theories R are suddenly homomorphic.

Theorem 3. *The overlay representations of Lie algebras form an overlay representation theory $\mathbf{O}(R)$, where R is a standard representation theory of Lie algebras (the covers of the Lie algebras are defined by the dense connected Lie composites and the target subcategory \mathcal{A}_0 consists of all Lie algebras $\text{End}(H)$).*

Note that the Grothendieck topology of the Theorem 3 differs from the usual one.

Remark 7.

- If R is a homomorphic representation theory for the ground category \mathcal{A} then for any object a of \mathcal{A} the category $\mathbf{C}(R)(a)$ is a subcategory of $\mathbf{O}(R)(a)$.
- $\mathbf{C}(\mathbf{O}(R)) = \mathbf{O}(R)$.

Problems:

- To formulate (if possible) the definition of the overlay representation theories $\mathbf{O}(R)$ for the nonhomomorphic representation theories R (cf. a comment after the Definition 3B).
- To give a sheaf-theoretic description of the overlay representation theories (cf. the same comment).
- To explicate the categorical or sheaf-theoretic relation between the composite and overlay representation theories.
- To formulate the general categorical version of the induction procedure (of the functors $\text{Ind}_a^b : R(a) \mapsto R(b)$, a is a subobject of the object b of the ground category) in the representation theory [5] and adapt it to the composite and overlay representation theories.
- To adapt (if possible) the Tannaka–Krein theory [5] (see also [8]) to the overlay representations.

Remark 8. For some ground categories \mathcal{A} and some representation theories R (e.g. finite dimensional representations of finite groups and reductive Lie algebras or unitary representations of compact groups) one may construct the Grothendieck group (ring) $\Gamma(R(a))$ of virtual representations (a is any object of \mathcal{A}) from the abelian (tensor) category $R(a)$ [5] and, therefore, the pre-sheaves $\Gamma\mathcal{R}$ of abelian groups (rings) over \mathcal{A} (note that the Grothendieck rings $\Gamma(R(\cdot))$ are supplied by numerous operations such as symmetric and exterior degrees, Adams operations and arbitrary “polynomial operations” related to the irreducible representations of the symmetric groups S_n [5]). The transition to the composite representation theory supply the topologized ground category \mathcal{A} by the sheaf $\mathbf{C}(\Gamma\mathcal{R})$ over it, so one may consider the cohomologies $H^*(\mathcal{A}, \mathbf{C}(\Gamma\mathcal{R}))$ of \mathcal{A} with coefficients in $\mathbf{C}(\Gamma\mathcal{R})$. The additional operations supply the cohomologies $H^*(\mathcal{A}, \mathbf{C}(\Gamma\mathcal{R}))$ by a sophisticated algebraic structure.

CONCLUSIONS

Thus, a categorical framework for the non-standard representation theories, which appear in the analysis of hidden symmetries [3,4], is briefly described. The relations of the abstract categorical constructions to the concrete inverse problems of the representation theory are explicated. Open questions are formulated. It seems that as the abstract categorical formalism as its appearance in the context of the analysis of hidden symmetries may be useful for the understanding of some general representation theoretic aspects of the control theory [9] and applications of the representation theory to the concrete problems of (classical and quantum) controlled systems as well as for many other subjects.

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